two solutions, if real, may be found by means of a circle on $\alpha\omega$ as diameter, as indicated in the figure.[1]

II. Supplementary Note on the Irrationality of Certain Trigonometric Functions.

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Theorem: *When an angle is rationally expressible in degrees and not a multiple of 30° or 45°, its trigonometric functions are irrational.*

The tangent of any rationally expressible angle except one of the form treated below may be proved irrational by a method so closely analogous to that used in my former paper that the details are omitted here. There remain angles of the form $(m/n)°$, where m/n is an irreducible fraction and where $m = k45$.

In the expansion

$$\tan n\theta = \frac{n \tan\theta - \dfrac{n(n-1)(n-2)}{3!}\tan^3\theta + \cdots \pm \tan^n\theta (\text{or} \pm n \tan^{n-1}\theta, \text{ when } n \text{ is even})}{1 - \dfrac{n(n-1)}{2!}\tan^2\theta + \cdots \pm n\tan^{n-1}\theta \ (\text{or } \tan^n\theta, \text{ when } n \text{ is even})}$$

the substitution $\tan n\theta = \pm 1$ gives an nth degree equation whose only possible rational roots (± 1) are excluded by the conditions of the problem. When $\tan n\theta = 0$ or $\pm\infty$, it is evident that the roots obtained by equating the numerator and denominator respectively to zero are either integers or the reciprocals of integers. Furthermore, if $\tan\theta$ is rational, $\tan 2\theta$, when it is finite, is rational. But if $\tan\theta = k$ or $1/k$, k being an integer, $\tan 2\theta = \pm 2k/(1-k^2)$, which is neither an integer nor the reciprocal of an integer, since the denominator lacks the factor k. Hence $\tan\theta$ is irrational.

The above theorem is interesting in that it brings out in a striking way for pedagogical purposes the great preponderance of irrational over rational numbers. Divide a right angle into n equal parts. Then there are only three distinct angles in all this infinite family which possess rational trigonometric functions.

Incidentally I have shown that the algebraic equations of the form (n odd)

$$x^{n-1} - \frac{n(n-1)}{2!}x^{n-3} + \cdots \pm n = 0,$$

and

$$nx^{n-1} - \frac{n(n-1)(n-2)}{3!}x^{n-3} + \cdots \pm 1 = 0,$$

have no rational roots.

[1] General Bixby would be glad to hear from any engineer, physicist or mathematician who is frequently required to solve higher equations, what method he uses, how much time is required, and what accuracy is deemed necessary.